

# One-Loop Quantum Energy Densities of Domain Wall Field Configurations

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## Abstract

We discuss a simple procedure for computing one-loop quantum energies of any static field configuration that depends non-trivially on only a single spatial coordinate. We specifically focus on domain wall-type field configurations that connect two distinct minima of the effective potential, and may or may not be the solutions of classical field equations. We avoid the conventional summation of zero-point energies, and instead exploit the relation between functional determinants and solutions of associated differential equations. This approach allows ultraviolet divergences to be easily isolated and extracted using any convenient regularization scheme. Two examples are considered: two-dimensional  $\phi^4$  theory, and three-dimensional scalar electrodynamics with spontaneous symmetry breaking at the one-loop level.

## I. INTRODUCTION

The calculation of the one-loop energy of an arbitrary static background field configuration is a problem in field theory that has received much attention over the years [1–5]. Various complications arise when one deals with field configurations with non-trivial spatial dependence, as extra care is required to relate renormalization of the effective action for such a field configuration to the renormalization used in the perturbative sector of a theory.

To illustrate this situation, let us recall the result for the kink mass in 1+1-dimensional  $\phi^4$  scalar field theory [1]. The typical way of computing one-loop corrections involves the summation over zero-point energies computed in the presence of a kink, and subsequent subtraction of the contribution from the vacuum sector. The necessary regularization is a rather delicate procedure, and it has been recognized long ago that the naive use of a momentum-cutoff regulator leads to an incorrect answer [1,6]. Of course, the fact that the one-loop calculation of the kink mass could be performed analytically is directly related to the special form of the kink solution for which the spectrum of the fluctuation operator in the kink background is exactly known.

For a generic background field configuration, the situation becomes more problematic as there is no general procedure for solving the eigenvalue problem analytically, and thus one must rely on numerical methods. This means that the sum over zero-point energies

must be regularized properly, and this regularization procedure must leave one with a well-defined regulator-independent integral accessible to numerical evaluation. The recent paper [4] gives one approach for providing an unambiguous regularization of the sum over zero-point energies, based on the subtraction of successive Born approximations for scattering phase shifts. Having rendered the zero-point sum finite by this subtraction, one may then add the subtracted terms back in a regularized form, and observe that the regulator dependence cancels appropriate counterterm contributions contained in the classical expression for the energy.

However, the approach of Ref. [4] can become complicated as one goes to higher dimensions (as higher order approximations for the phase shifts are needed), or encounter fluctuation operators that couple multiple fluctuating fields, as one inevitably does in gauge theories. It is therefore desirable to have a method that would allow a more computationally straightforward treatment of ultraviolet divergences.

We will discuss an alternative method, specifically applicable to kink-type field configurations that depend non-trivially on only a single spatial coordinate, which naturally simplifies the treatment of ultraviolet divergences and also allows one to deal with the other problems that appear in the treatment of gauge theories.

The crucial feature of the method we describe is the departure from the conventional summation over zero-point energies. Instead, we go back to the functional determinants that enter the one-loop effective action, and express them in terms of the solutions to associated differential equations with certain boundary conditions. We examine two specific examples to illustrate the main ideas. One natural choice is, of course, the classic case of 1+1-dimensional scalar  $\phi^4$  field theory. This simple case provides us with a consistency check, as the value of kink mass can be computed independently by an analytic calculation. In addition, working with two-dimensional  $\phi^4$  theory presents an opportunity to illustrate the relevant ideas in a simple context.

We then examine three-dimensional scalar electrodynamics. This theory has a first order phase transition (as the scalar mass is varied) separating Coulomb and Higgs phases. But the existence of a first order transition, and hence the presence of a degenerate ground state, is only apparent when one-loop quantum corrections are included. Hence, computing the interface tension (or domain wall energy density) at the phase transition requires the evaluation of the effective action of field configurations which are not classical solutions of the underlying classical field theory. To our knowledge, our approach provides the first practical technique for computing one-loop energies of arbitrary Higgs field profiles with one-dimensional spatial dependence in a gauge theory.

Three-dimensional scalar QED can be viewed as a low energy effective theory for four-dimensional scalar QED at finite temperature. The domain wall surface tension in the three-dimensional theory translates into the surface tension of an interface that separates hot and cold phases of the finite temperature theory. This quantity has been a subject of several investigations in electroweak theory [7–9], and scalar electrodynamics has often been used as a toy model [10]. The desire to have a way of correctly computing the full one-loop interface tension in electroweak theory was part of the motivation for this work.

This paper is organized as follows. In the next section, we consider two-dimensional scalar  $\phi^4$  theory. The results of this section are not new, but this toy model provides an useful playground for setting up notation, introducing the basic ideas of our method, identifying

possible problems, and developing appropriate tools for solving them.

In section 3, we consider three-dimensional scalar QED. We show how to deal with the difficulties associated with the absence of degeneracy of the ground state at the tree level. We also show how the scheme used to compute one-dimensional determinants in section 2 may be extended to deal with a fluctuation operator that couples together more than one field. Numerical results are presented at the end of this section, where we show that they are consistent with expected behavior in the limit of a small ratio of scalar and gauge couplings.

## II. TWO-DIMENSIONAL $\phi^4$ THEORY

In this section we will be concerned with our toy model, double well scalar  $\phi^4$  field theory in two dimensions. We start by formulating the theory and introducing our notation. We then review the traditional way of calculating one-loop corrections to the energy of an arbitrary field configuration<sup>1</sup> by summing over zero-point energies. We implement this prescription and compute the one-loop correction to the kink mass using dimensional regularization. This warm-up computation, which reproduces old results, will serve to illustrate the difficulties that one faces in generalizing this approach to arbitrary field configurations and more complicated theories. We will argue that the integration over one of the directions (Euclidean time), which reduces the functional determinant to a sum over zero-point energies, does not come without a price. As we will see, this approach forces one to resort to computationally inconvenient ways of regulating ultraviolet divergences.

We then introduce an alternative method that does not rely on the summation of zero-point energies. We show how one can make the issue of ultraviolet regularization simple by integrating over the momentum in the direction of spatial variation of the field profile, instead of integrating over the Euclidean frequency. We relate the resulting one-dimensional functional determinants to solutions of associated differential equations, and present an explicit formula for the mass of an arbitrary field configuration that connects the two minima of the effective potential. The value of the one-loop correction to the kink mass is reproduced as a check of this general result.

### A. One-loop energy density

Consider scalar  $\phi^4$ -theory with the double well potential

$$V(\phi) = -\frac{m_0^2}{2}\phi^2 + \frac{\lambda_0}{4!}\phi^4, \quad (2.1)$$

which exhibits spontaneous symmetry breaking at the tree level. This feature persists quantum-mechanically due to the  $\phi \rightarrow -\phi$  symmetry.

We consider a Euclidean version of this theory formulated inside a finite  $n$ -dimensional box with Dirichlet boundary conditions for the fluctuating fields. Formulating a theory in a

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<sup>1</sup>In 1+1 dimensions what would be the “energy density” of a domain wall in higher dimensions becomes just the energy, or mass, of the kink-like field configuration.

variable number of dimensions allows us to use dimensional regularization, even though the final results of this section will only apply to the two-dimensional theory.

We will be dealing with field configurations that vary only along a single direction. We will assign the subscript  $\perp$  to this direction, and the relevant coordinate will be denoted by  $x_\perp$ . The coordinates along all other directions are denoted by  $x_\parallel$ . We will refer to these directions as “worldvolume” coordinates in obvious analogy with D-branes in string theory.

We will denote by  $L$  the half-length of the box in the direction parameterized by  $x_\perp$ , and by  $V_\parallel$  the volume of an  $(n-1)$ -dimensional slice  $x_\perp = \text{const}$ . Both  $L$  and  $V_\parallel$  will be taken to infinity at the end of the calculation, leaving us with the theory in infinite Euclidean space.

We further restrict our attention to field configurations  $h(x_\perp)$  that obey the following boundary conditions:

$$h(-L) = -v, \quad (2.2)$$

$$h(+L) = +v, \quad (2.3)$$

where  $\pm v = \langle \phi \rangle$  are the minima of the effective potential. The tree-level value of  $v$  is given by  $v_{\text{tree}}^2 = \frac{6m^2}{\lambda_0}$ . We require the field profile to approach the limiting values at  $x_\perp = \pm\infty$  fast enough so that the energy density  $\sigma$ , which is related to the effective action  $\Gamma[h]$  as

$$\sigma = \lim_{V_\parallel, L \rightarrow \infty} \frac{\Gamma[h]}{V_\parallel}, \quad (2.4)$$

is finite.<sup>2</sup>

The effective action can be evaluated perturbatively and is equal to the classical action evaluated at the given field profile plus quantum corrections. Retaining only one-loop corrections, one has the following expression for  $\sigma$ :

$$\sigma = \int dx_\perp [\mathcal{L}(h(x_\perp)) - \mathcal{L}(v)] + \lim_{V_\parallel, L \rightarrow \infty} \left[ \frac{1}{2V_\parallel} \ln \det \frac{\Delta}{\Delta^{(0)}} \right], \quad (2.5)$$

where  $\Delta = -\partial_\perp^2 - \partial_\parallel^2 + V''$  is the fluctuation operator in the non-zero background field sector, while  $\Delta^{(0)}$  is the corresponding fluctuation operator in the vacuum sector.

We can split  $\sigma$  into three parts that correspond, respectively, to the piece of the classical energy density that contains only renormalized parameters  $m$  and  $\lambda$ , the one-loop quantum corrections, and the counterterm contribution to the classical energy density,

$$\sigma = \sigma_{\text{cl}} + \sigma_{\text{qu}} + \sigma_{\text{c.t.}} \quad (2.6)$$

The classical piece  $\sigma_{\text{cl}}$ , and the quantum correction combined with the counterterm contribution,  $\sigma_{\text{qu}} + \sigma_{\text{c.t.}}$ , will each be UV-finite.

The terms that enter (2.6) are given by

$$\sigma_{\text{cl}} = \int dx_\perp \left[ \frac{1}{2} (\partial_\perp h(x_\perp))^2 - \frac{m^2}{2} h(x_\perp)^2 + \frac{\lambda}{4!} h(x_\perp)^4 + \frac{3m^4}{2\lambda} \right], \quad (2.7)$$

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<sup>2</sup>We will always normalize the effective action so that it vanishes for vacuum field configurations:  $\Gamma[h \equiv \pm v] = 0$ .

$$\sigma_{\text{qu}} = \lim_{L, V_{\parallel} \rightarrow \infty} \left[ \frac{1}{2V_{\parallel}} \text{tr} \ln \left( \frac{-\partial_{\perp}^2 - \partial_{\parallel}^2 + V''}{-\partial_{\perp}^2 - \partial_{\parallel}^2 + V^{(0)''}} \right) \right], \quad (2.8)$$

and

$$\sigma_{c.t.} = -\frac{m_0^2 - m^2}{2} \int dx_{\perp} \left[ h(x_{\perp})^2 - \frac{6m^2}{\lambda} \right]. \quad (2.9)$$

In the Eq. (2.8),  $V^{(0)''} = 2m^2$  denotes the tree-level curvature of the effective potential at the stationary points.

The fluctuation operators are  $x_{\parallel}$ -independent and can therefore be partially diagonalized by a Fourier transform over the  $x_{\parallel}$  coordinate. Taking the  $V_{\parallel} \rightarrow \infty$  limit recasts  $\sigma_{\text{qu}}$  as

$$\sigma_{\text{qu}} = \lim_{L \rightarrow \infty} \left[ \frac{1}{2} \int \frac{d^{n-1}k_{\parallel}}{(2\pi)^{n-1}} \ln \det_{\perp} \left( \frac{k_{\parallel}^2 - \partial_{\perp}^2 + V''}{k_{\parallel}^2 - \partial_{\perp}^2 + V^{(0)''}} \right) \right], \quad (2.10)$$

where  $\det_{\perp}$  is a functional determinant of a linear operator acting on the Hilbert space  $\mathcal{H}_L$  of functions of  $x_{\perp}$  that vanish at  $\pm L$ .

## B. Summing over zero point energies

The conventional approach to computing the one-loop energy relies on making (2.10) into a sum of zero-point energies. To accomplish this, one chooses one of the worldvolume directions to play the role of Euclidean time, and integrates over the corresponding component of  $k_{\parallel}$ . This allows one to rewrite  $\sigma_{\text{qu}}$  in terms of the eigenvalues  $\epsilon_i^2$  of the operator  $-\partial_{\perp}^2 + V''$ ,

$$\sigma_{\text{qu}} = \frac{1}{2} \int \frac{d^{n-2}k_{\parallel}}{(2\pi)^{n-2}} \sum_i \left[ \sqrt{\vec{k}_{\parallel}^2 + (\epsilon_i)^2} - \sqrt{\vec{k}_{\parallel}^2 + (\epsilon_i^{(0)})^2} \right]. \quad (2.11)$$

This expression requires the presence of some regulator which will render finite both the sum and the integration over the remaining worldvolume directions. In the two-dimensional case, if we don't use dimensional regularization, then there are no integrations left to do and the expression for  $\sigma_{\text{qu}}$  becomes just a suitably regularized sum over zero-point energies,

$$\sigma_{\text{qu}} = \frac{1}{2} \sum_i \left[ \epsilon_i - \epsilon_i^{(0)} \right]. \quad (2.12)$$

It is expression (2.12) that has conventionally been used for computing the one-loop energies in 1+1 dimensional scalar field theories [1,4]. Below we will re-evaluate the one-loop correction (2.11) to the mass of a kink in the scalar 1+1  $\phi^4$  theory using dimensional regularization. The answer has been known for quite a long time [1], but it is interesting to see how dimensional regularization can naturally avoid various regularization-related subtleties which appear in more traditional treatments [1].

The primary purpose of this computation, however, is to illustrate that the direct use of dimensional regularization applied to (2.11) (without prior subtraction of Born approximations as in [4]) for an arbitrary field configuration is not desirable. The computation that

we are about to present is feasible only because the fluctuation operator for this particular field profile is nice enough that its spectrum can be obtained analytically.

Let us again go back to the theory formulated in  $n$  dimensions, keeping in mind that physical results will be obtained via the limit  $n \rightarrow 2$ . The quartic coupling is only multiplicatively renormalized to one-loop order:

$$\lambda_0 = \mu^{2-n} \lambda, \quad (2.13)$$

with  $\lambda$  bearing the dimension of mass squared; its beta function vanishes in the physical limit  $n \rightarrow 2$ .

The relation between the  $\mu$ -dependent renormalized parameter  $m(\mu)$  and the bare parameter  $m_0$  can be fixed by a calculation of the one-loop correction to the self-energy in the perturbative sector of a theory (which is defined with respect to one of the degenerate ground states). This correction has a pole at  $n = 2$ . To render the self-energy finite, the counterterm must also have a pole with the same residue. One has, of course, the freedom to add some finite quantity to the counterterm. We exploit this freedom by choosing  $\overline{MS}$ -subtraction, which in the two-dimensional theory is defined as

$$m_0^2 = m(\mu)^2 - \mu^{n-2} \frac{\lambda_0}{8\pi} \left( \frac{e^\gamma}{4\pi} \right)^{n/2-1} \frac{1}{n/2-1}, \quad (2.14)$$

where  $\gamma$  is Euler's constant.

It is easy to determine the one-loop beta function of  $m(\mu)^2$ :

$$\mu \frac{d}{d\mu} m^2(\mu) = \frac{\lambda}{4\pi} + \mathcal{O}(\lambda^2). \quad (2.15)$$

To calculate the kink mass we set the background field configuration  $h(x_\perp)$  equal to the classical kink solution,

$$h(x_\perp) = m_0 \sqrt{6/\lambda_0} \tanh \left( \frac{mx_\perp}{\sqrt{2}} \right). \quad (2.16)$$

As in (2.6), the kink mass  $\sigma^{(\text{kink})}$  can be split into classical, quantum, and counterterm contributions:

$$\sigma^{(\text{kink})} = \sigma_{\text{cl}}^{(\text{kink})} + \sigma_{\text{qu}}^{(\text{kink})} + \sigma_{\text{c.t.}}^{(\text{kink})}. \quad (2.17)$$

By direct evaluation of (2.7), the renormalized part of the classical contribution, which we conventionally denote by  $\sigma_{\text{cl}}^{(\text{kink})}$ , is given by

$$\sigma_{\text{cl}}^{(\text{kink})} = 4\sqrt{2} \frac{m^3(\mu)}{\lambda_0}, \quad (2.18)$$

while the explicit form of the counterterm contribution (2.9) is

$$\sigma_{\text{c.t.}}^{(\text{kink})} = -\frac{3\sqrt{2}}{4\pi} m \left( \frac{e^\gamma}{4\pi} \right)^{n/2-1} \frac{\mu^{n-2}}{n/2-1}. \quad (2.19)$$

The quantum piece  $\sigma_{\text{qu}}^{(\text{kink})}$  can be calculated using (2.11) and the spectrum of the operator  $-\partial_{\perp}^2 + V''$ . Since we are using dimensional regularization, we can formally rewrite the sum in (2.11) as an integral,

$$\sigma_{\text{qu}}^{(\text{kink})} = \frac{1}{2} \int \frac{d^{n-2}k_{\parallel}}{(2\pi)^{n-2}} \left[ (k_{\parallel}^2)^{1/2} + (k_{\parallel}^2 + \frac{3}{2}m^2)^{1/2} + \int \frac{dk_{\perp}}{(2\pi)} (k_{\parallel}^2 + k_{\perp}^2 + 2m^2)^{1/2} \delta'(k_{\perp}) \right]. \quad (2.20)$$

Here  $\delta(k_{\perp})$  is the scattering phase shift and the explicit form [1] of  $\delta'(k_{\perp}) \equiv \frac{d\delta(k_{\perp})}{dk_{\perp}}$  is

$$\delta'(k_{\perp}) = -6\sqrt{2}m \frac{(k_{\perp}^2 + m^2)}{(2k_{\perp}^2 + m^2)(k_{\perp}^2 + 2m^2)}. \quad (2.21)$$

The first two terms in (2.20) come from the two bound states of the small fluctuation operator in the kink background, and the last term from the continuous part of the spectrum [1]. The last term involves a difference between the contributions from the kink and vacuum sectors, which do not cancel because of the differing spectral densities in these sectors.

Substituting (2.21) into (2.20) one arrives at the following expression for  $\sigma_{\text{qu}}^{(\text{kink})}$ ,

$$\begin{aligned} \sigma_{\text{qu}}^{(\text{kink})} = & \frac{1}{2} \int \frac{d^{n-2}k_{\parallel}}{(2\pi)^{n-2}} \left[ (k_{\parallel}^2)^{1/2} + (k_{\parallel}^2 + \frac{3}{2}m^2)^{1/2} - 3\sqrt{2}m \int \frac{dk_{\perp}}{2\pi} \frac{m^2(k_{\parallel}^2 + k_{\perp}^2 + 2m^2)^{1/2}}{(2k_{\perp}^2 + m^2)(k_{\perp}^2 + 2m^2)} \right. \\ & - 3\sqrt{2}m \int \frac{dk_{\perp}}{2\pi} \frac{k_{\parallel}^2}{(k_{\parallel}^2 + k_{\perp}^2 + 2m^2)^{1/2}(k_{\perp}^2 + 2m^2)} \\ & \left. - 3\sqrt{2}m \int \frac{dk_{\perp}}{2\pi} \frac{1}{(k_{\parallel}^2 + k_{\perp}^2 + 2m^2)^{1/2}} \right]. \quad (2.22) \end{aligned}$$

Let us analyze (2.22), term by term. The first term that involves  $(k_{\parallel}^2)^{1/2}$  is set to zero by dimensional regularization, as it does not contain any external dimensionfull parameters. The second, third, and fourth terms are finite, and equal  $3m/2\sqrt{6}$ ,  $-m/\sqrt{6}$ , and  $-3m/\sqrt{2}\pi$  respectively. One way to obtain these results is to formally rewrite each of the corresponding  $(n-2)$ -dimensional integrals as a product of an angular integral which is equal to  $2\pi^{n/2-1}/\Gamma(n/2-1)$  and a radial integral which can be shown to contain a  $(n/2-1)^{-1}$  pole that cancels the zero in inverse Gamma function, leading to a finite expression in the limit  $n \rightarrow 2$ . The last term in (2.22) can be computed along the same lines,

$$-\frac{3\sqrt{2}m}{2} \int \frac{d^{n-2}k_{\parallel}}{(2\pi)^{n-2}} \int \frac{dk_{\perp}}{2\pi} \frac{1}{(k_{\parallel}^2 + k_{\perp}^2 + 2m^2)^{1/2}} = -\frac{3\sqrt{2}m}{4\pi} \left( \frac{2m^2}{4\pi\mu^2} \right)^{n/2-1} \mu^{n-2} \Gamma(1-n/2). \quad (2.23)$$

It obviously produces a pole that cancels the pole in  $\sigma_{\text{c.t.}}^{(\text{kink})}$ , leaving logarithmic dependence on the scale  $\mu$ . This is completely analogous to what happens when one does conventional  $\overline{MS}$  regularization in the perturbative sector.

We are now in position to collect all the terms together, and remove the UV regulator by taking the  $n \rightarrow 2$  limit. This leads to the following expression for the kink mass:

$$\sigma^{(\text{kink})} = 4\sqrt{2} \frac{m^3(\mu)}{\lambda} + \left[ \frac{1}{2\sqrt{6}} - \frac{3}{\sqrt{2}\pi} \right] m + \frac{3\sqrt{2}}{4\pi} \ln(2m^2/\mu^2). \quad (2.24)$$

One can check that this expression is  $\mu$ -independent to  $\mathcal{O}(\lambda)$ .

Finally, to make a connection to previous results in the literature, we must establish a relation between the  $\overline{MS}$  mass  $m(\mu)$  and other  $\mu$ -independent physical quantities. The renormalized mass  $M$  used in Ref. [1] was defined by the condition that the one-loop effective potential has its minima fixed at  $\langle\phi\rangle = \pm(6M^2/\lambda)^{1/2}$ . This corresponds to the complete cancellation of the tadpole diagrams by the counterterm. The relation between  $M$  and  $m(\mu)$  to one-loop order is given by

$$M^2 = m^2(\mu) + \frac{\lambda}{8\pi} \ln(2m^2/\mu^2). \quad (2.25)$$

Substituting this into Eq. (2.20) gives

$$\sigma^{(\text{kink})} = 4\sqrt{2}\frac{M^3}{\lambda} + \left[ \frac{1}{2\sqrt{6}} - \frac{3}{\sqrt{2\pi}} \right] M. \quad (2.26)$$

which coincides with the well-known result for the kink mass of Ref. [1].

The most important lesson that should be learned from this calculation is that it would be very difficult to generalize this dimensional continuation approach to the case of an arbitrary field configuration. Indeed, the discussion below (2.22) shows that even after  $\sigma_{\text{c.t.}}$  cancels the pole in the quantum part of  $\sigma$ , taking the  $n \rightarrow 2$  limit is still a bit subtle. Namely, it involves the cancellation of a zero from the angular integral with a pole from the analytical continuation of the radial integral. This procedure would be hard to implement in general, as the radial integral for an arbitrary field configuration can only be evaluated numerically. Therefore one must consider other ways of regulating a theory, or abandon the zero-point energy approach. The former option is chosen in [4], while the latter choice is going to be explored below.

### C. Integrating over $k_\perp$

One may compute the one-dimensional functional determinant in (2.10) directly, without converting it to a sum over zero point energies. The benefit, as we have already advertised, will be the ability to reduce the computation of the quantum correction to the energy of an arbitrary field profile,  $\sigma_{\text{qu}} + \sigma_{\text{c.t.}}$ , to a finite integral that can be evaluated numerically.

It should be clear that the only reason for this procedure not being completely trivial is the existence of ultraviolet divergences in the theory. Our aim is the establishment of a regulator-independent method, which clearly separates the divergences and allows one to use any conventional regularization scheme in dealing with them.

We will start the analysis of the ultraviolet divergences by considering an expansion of functional determinant in (2.8) in powers of  $(\Delta - \Delta^{(0)})/\Delta^{(0)}$ :

$$\sigma_{\text{qu}} = \lim_{V_\parallel, L \rightarrow \infty} \left[ \frac{1}{2V_\parallel} \text{tr} \sum_n \frac{(-)^n}{n} \left( \frac{\Delta - \Delta^{(0)}}{\Delta^{(0)}} \right)^n \right]. \quad (2.27)$$

In two dimensions, only the first term in this expansion is divergent. Recall that  $\Delta^{(0)}$  is just an inverse bare propagator, so this divergent term can be written as



$$\lim_{V_{\parallel}, L \rightarrow \infty} \left( \frac{1}{2V_{\parallel}} \text{tr} \left[ (V'' - 2m^2)(-\nabla^2 + 2m^2)^{-1} \right] \right) = \frac{F}{2} \int \frac{d^2 k}{(2\pi)^2} \frac{1}{k^2 + 2m^2}, \quad (2.28)$$

where  $F$  is the Fourier transform of  $(V'' - 2m^2)$  taken at zero momentum,

$$F = \int dx_{\perp} [(V'' - 2m^2)] = \frac{\lambda}{2} \int dx_{\perp} \left[ h(x_{\perp})^2 - \frac{6m^2}{\lambda} \right]. \quad (2.29)$$

Of course, the integral in (2.28) is logarithmically divergent. However, we are free to integrate over the momentum in the direction perpendicular to the worldvolume and then impose any convenient cutoff on the remaining integral.<sup>3</sup> Subtracting this divergent term from (2.10), and adding it back, we obtain the following expression for the finite quantity  $\sigma_{\text{qu}} + \sigma_{\text{c.t.}}$ :

$$\begin{aligned} \sigma_{\text{qu}} + \sigma_{\text{c.t.}} = & \frac{1}{2} \int \frac{dk_{\parallel}}{2\pi} \left[ \lim_{L \rightarrow \infty} \ln \frac{\det_{\perp} (k_{\parallel}^2 - \partial_{\perp}^2 + V'')}{\det_{\perp} (k_{\parallel}^2 - \partial_{\perp}^2 + 2m^2)} - \frac{F}{2(k_{\parallel}^2 + 2m^2)^{1/2}} \right] \\ & + \frac{F}{4} \int^{\Lambda} \frac{dk_{\parallel}}{2\pi} \frac{1}{(k_{\parallel}^2 + 2m^2)^{1/2}} + \sigma_{\text{c.t.}}. \end{aligned} \quad (2.30)$$

The first integral in this expression is finite, as the divergent part has been subtracted from it. The second term in (2.30) contains an integral that appears in the evaluation of the tadpole diagram, and needs to be regulated. We indicate this by writing  $\Lambda$ , which should be thought of as some cutoff on worldvolume momentum, at the top of this integral. However we can use any regulator we like, as long as it allows unbounded integration over  $k_{\perp}$ . The same tadpole diagram also contributes to the mass renormalization in the perturbative sector, and consecutively, appears in  $\sigma_{\text{c.t.}}$ . Comparing (2.9) and (2.29) one can immediately see that the UV-divergent terms in (2.30) cancel each other, leaving a finite remainder whose value depends on the renormalization conditions.

Thus, the only non-trivial piece of the calculation is the evaluation of a single one-dimensional integral. The expression (2.30) is, therefore, a key result of this section. Note, that had the number of dimensions  $n$  been greater than two, more subtractions might have been needed. The structure of such terms would be quite clear, though, as it is determined by the perturbative expansion (2.27).

In what follows we are going to regulate the UV divergences with dimensional regularization where the number of worldvolume dimensions  $n-1$  is regarded as a variable parameter. Since the divergent terms in (2.30) are isolated, the treatment that is involved is essentially the same as appeared when we considered the analytic kink solution. We choose to express our result in terms of the physical parameter  $M$  defined by (2.25). One finds simply

$$\sigma_{\text{qu}} + \sigma_{\text{c.t.}} = \frac{1}{2} \int \frac{dk_{\parallel}}{2\pi} \left[ \lim_{L \rightarrow \infty} \ln \frac{\det_{\perp} (k_{\parallel}^2 - \partial_{\perp}^2 + V'')}{\det_{\perp} (k_{\parallel}^2 - \partial_{\perp}^2 + 2M^2)} - \frac{F}{2(k_{\parallel}^2 + 2M^2)^{1/2}} \right]. \quad (2.31)$$

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<sup>3</sup>In contrast, the zero-point energy approach does not allow regulator-independent integration over  $k_{\perp}$ , as an integration over one of the components of  $k_{\parallel}$  has already been performed.

When the field configuration is a solution of the classical equations of motion, the operator  $-\partial_\perp^2 + V''$  acquires a zero eigenvalue, associated with the spontaneous breaking of translational invariance. Consecutively, the 2-dimensional fluctuation operator  $-\partial_\parallel^2 - \partial_\perp^2 + V''$  has continuous spectra extending down to zero. In principle one should go back to the original functional integral and use standard collective coordinate procedures [11] to separate the translational mode. However, the zero mode of  $-\partial_\perp^2 + V''$  merely causes the two-dimensional functional determinant appearing in (2.31) to have an integrable logarithmic singularity at  $k_\parallel = 0$ . Therefore, we do not need to modify (2.31) for the computation of the one-loop kink mass.

To proceed further, one needs to evaluate the ratio of one-dimensional functional determinants appearing in (2.31). Fortunately, we can use the result of [12] where this ratio is directly related to solutions of associated differential equations. (We review this construction in the appendix.) Specifically, this ratio of one-dimensional determinants can be written as

$$\frac{\det_\perp (k_\parallel^2 - \partial_\perp^2 + V'')}{\det_\perp (k_\parallel^2 - \partial_\perp^2 + 2M^2)} = \frac{\psi(L)}{\psi^{(0)}(L)}, \quad (2.32)$$

where the  $\psi$ 's are the solutions to the associated differential equations,

$$[k_\parallel^2 - \partial_\perp^2 + V''] \psi(x_\perp) = 0, \quad (2.33)$$

and

$$[k_\parallel^2 - \partial_\perp^2 + 2M^2] \psi^{(0)}(x_\perp) = 0, \quad (2.34)$$

with initial conditions

$$\psi(-L) = 0; \quad \psi(-L)' = 1, \quad (2.35)$$

and

$$\psi^{(0)}(-L) = 0; \quad \psi^{(0)}(-L)' = 1. \quad (2.36)$$

The free solution  $\psi^{(0)}(x_\perp)$  is trivial to find. It is given by

$$\psi^{(0)}(x_\perp) = \frac{\sinh [(k_\parallel^2 + 2M^2)^{1/2}(L + x_\perp)]}{(k_\parallel^2 + 2M^2)^{1/2}}. \quad (2.37)$$

Neglecting terms which are exponentially small as  $L \rightarrow \infty$ , the contribution of this solution to (2.32) is

$$\psi^{(0)}(L) = \frac{e^{-(k_\parallel^2 + 2M^2)^{1/2}2L}}{2(k_\parallel^2 + 2M^2)^{1/2}}. \quad (2.38)$$

It is desirable to extract explicitly the large  $L$  asymptotic behavior from  $\psi(x_\perp)$  as well. This prompts the natural substitution

$$\psi(x_\perp) = f(x_\perp) e^{-(k_\parallel^2 + 2M^2)^{1/2}(x_\perp + L)}, \quad (2.39)$$

and converts the ODE (2.33) to the corresponding equation for  $f(x_\perp)$ :

$$\left[-\partial_\perp^2 + 2(k_\parallel^2 + 2M^2)^{1/2}\partial_\perp + (V'' - 2M^2)\right] f(x_\perp) = 0, \quad (2.40)$$

with  $f(x_\perp)$  being subject to the same initial conditions,

$$f(-L) = 0; \quad f'(-L) = 1. \quad (2.41)$$

We can now present the result for the quantum correction to the mass of an arbitrary kink-like field configuration in two-dimensional  $\phi^4$  theory in its final form,

$$\sigma_{\text{qu}} + \sigma_{\text{c.t.}} = \frac{1}{2} \int \frac{dk_\parallel}{(2\pi)} \left[ \lim_{L \rightarrow \infty} \ln \left[ 2(k_\parallel^2 + 2M^2)^{1/2} f(L) \right] - \frac{F}{2(k_\parallel^2 + 2M^2)^{1/2}} \right]. \quad (2.42)$$

As a check of our method, we have used (2.40)–(2.42) to recalculate the value of  $\sigma_{\text{qu}} + \sigma_{\text{c.t.}}$  for the kink configuration, where the exact result is given by (see Eq. (2.26))

$$\sigma_{\text{qu}}^{(\text{kink})} + \sigma_{\text{c.t.}}^{(\text{kink})} = \left[ \frac{1}{2\sqrt{6}} - \frac{3}{\sqrt{2\pi}} \right] M. \quad (2.43)$$

Using the canned routines of *Mathematica* to integrate (2.40)–(2.42), one may easily reproduce this result with an accuracy better than one part in  $10^5$ .

### III. THREE-DIMENSIONAL SCALAR ELECTRODYNAMICS

We are now ready to tackle the gauge field theory case. For the rest of the paper we will be developing a method for computing one-loop energy densities (which will sometimes be called the surface or interface tension) of domain wall-type Higgs field configurations in three-dimensional scalar electrodynamics. The parameters of this theory will be tuned to sit at the first order phase transition where, at one-loop level, there is a degeneracy of the ground state, and consequently the existence of stable domain wall-type configurations. .

As we will see, the method that we applied in the previous section can be successfully generalized to scalar QED. However, there are two new problems which we did not encounter in the scalar theory. One is the existence of quadratic couplings between different fluctuating fields in the Lagrangian. This requires generalization of the result for a one-dimensional functional determinant. Another issue is the absence of tree-level degeneracy of the effective potential which will cause the quantum and classical parts of the surface tension to individually diverge with increasing box size. We will show how these problems can be resolved.

#### A. Formulation of the problem

We will start by formulating the theory and discussing the relations between the parameters that are required to tune the theory to its phase transition. We will see that the parameter that determines the validity of perturbation theory is the ratio of scalar and gauge couplings. In the regime where this parameter is small, it is in fact possible to analytically

compute the leading behavior of the interface tension. We will later compare the results of this analysis with the results of our full one-loop computations.

We consider the following Euclidean action for three-dimensional scalar electrodynamics,

$$S = \int d^3x \left[ \frac{1}{4} F_{ij} F_{ij} + \frac{1}{2} (D_i \Phi)^\dagger (D_i \Phi) + V(\Phi) + \mathcal{L}_{\text{g.f.}} + \mathcal{L}_{\text{gh}} + \mathcal{L}_{\text{c.t.}} \right] \quad (3.1)$$

where  $D_i \equiv \partial_i + igA_i$  denotes the usual covariant derivative with respect to the abelian gauge field  $A_i$ , and  $\mathcal{L}_{\text{g.f.}}$  and  $\mathcal{L}_{\text{gh}}$  denote gauge fixing and ghost contributions to the Lagrangian, respectively, which will be specified later. We will eventually be using dimensional regularization, so the counterterm Lagrangian  $\mathcal{L}_{\text{c.t.}}$  would in general contain  $(n-3)^{-1}$  poles where  $n$  is the analytically continued number of dimensions. In three-dimensional scalar QED, however, no poles appear at one-loop order. Hence, we may drop explicit mention of counterterms in all subsequent calculations.

We choose the simple quartic potential for the complex Higgs field  $\Phi$ ,

$$V(\Phi) = \frac{m^2}{2} |\Phi|^2 + \frac{\lambda}{4!} |\Phi|^4, \quad (3.2)$$

with a positive coefficient  $m^2$  in front of the quadratic term.

Let us briefly sketch the relations between the parameters of the theory that follow from the condition of a doubly degenerate ground state. Recall that because of the  $U(1)$  gauge symmetry the effective potential  $V_{\text{eff}}$  may be chosen to be a function of a real expectation value of the Higgs field. The crucial observation is that for the theory at hand, the effective potential acquires a  $-g^3 \Phi^3$  term which comes from the summation of all one-loop gauge field diagrams with arbitrary number of insertions of  $g^2 \Phi^2$  [13]. The existence of two degenerate minima of the effective potential hence implies that the effective potential acquires the form of a double well,

$$V_{\text{eff}} \approx \frac{m^2}{2v^2} \Phi^2 (\Phi - v)^2, \quad (3.3)$$

where  $v$  is the expectation value of  $\Phi$  in the Higgs phase. This immediately implies the following characteristic relations between the parameters:

$$\begin{aligned} v &\sim g^3/\lambda, \\ m^2 &\sim g^6/\lambda. \end{aligned} \quad (3.4)$$

Given (3.4), one can verify that the parameter controlling the reliability of perturbation theory both in symmetric and Higgs phases is  $\lambda/g^2$  [13]. For example, adding an extra loop in a diagram in the Higgs phase due to the addition of a gauge field propagator leads to an additional factor of order  $g^2 \int d^3k (k^2 + M_W^2)^{-2} \sim g^2/M_W \sim \lambda/g^2$ , where  $M_W = gv$  is the mass of a gauge boson. Thus, perturbation theory is reliable for small  $\lambda/g^2$ , even though the existence of a Higgs phase is not apparent at tree level, and is caused only by quantum corrections.

As in the previous section, the computation of the one-loop surface tension amounts to the evaluation of the one-loop effective action for a given field profile  $h \equiv h(x_\perp)$ , which we may choose to be real and positive. We will restrict our attention to domain wall-type configurations that obey boundary conditions similar to (2.2):

$$h(-L) = 0, \quad (3.5)$$

$$h(L) = v. \quad (3.6)$$

We put the system into a three-dimensional box of half-length  $L$  and transverse area  $V_{\parallel}$ . We decompose the complex Higgs field  $\Phi$  into the background field  $h$  plus two real fluctuating fields,

$$\Phi = h + \chi + i\phi, \quad (3.7)$$

and attribute gauge variations to the fluctuating fields  $\chi$  and  $\phi$ .

The effective action is a sum of classical and one-loop terms:

$$\Gamma[h] = \Gamma_{\text{cl}}[h] + \Gamma_{\text{qu}}[h], \quad (3.8)$$

where

$$\Gamma_{\text{cl}}[h] = V_{\parallel} \int dx_{\perp} \left[ \frac{1}{2} (\partial_{\perp} h(x_{\perp}))^2 + \frac{m^2}{2} h(x_{\perp})^2 + \frac{\lambda}{4!} h(x_{\perp})^4 \right]. \quad (3.9)$$

To get the quantum part, we need to expand the action (3.1) around the background field configuration, keeping terms that are quadratic in fluctuating fields. In doing so, one notices that a term of the form  $A_i \partial_i \phi$  appears. In the presence of a spatially varying background field, it is not possible to entirely eliminate this cross-coupling term. But one may partly eliminate this term by choosing a modification of  $R_{\xi}$  gauge with the following gauge-fixing term,

$$\mathcal{L}_{\text{g.f.}} = \frac{1}{2\xi} \left[ \partial_i A_i + \xi \frac{g}{2} \text{Im}(\Phi^2) \right]^2. \quad (3.10)$$

The part of the action quadratic in fluctuating fields then reads

$$S^{(2)} = \int d^3x (\mathcal{L}_{\chi} + \mathcal{L}_A + \mathcal{L}_{\phi} + \mathcal{L}_{A\phi} + \mathcal{L}_{\text{gh}}), \quad (3.11)$$

where

$$\mathcal{L}_{\chi} = \frac{1}{2} (\partial_i \chi)^2 + \left[ \frac{m^2}{2} + \frac{\lambda}{4} h(x_{\perp})^2 \right] \chi^2, \quad (3.12)$$

$$\mathcal{L}_A = \frac{1}{2} A_i \left( -\delta_{ij} \partial^2 - (1 - \xi^{-1}) \partial_i \partial_j + g^2 h(x_{\perp})^2 \delta_{ij} \right) A_j, \quad (3.13)$$

$$\mathcal{L}_{\phi} = \frac{1}{2} (\partial_i \phi)^2 + \left( \frac{m^2}{2} + \left[ \frac{\lambda}{12} + \frac{\xi}{2} g^2 \right] h(x_{\perp})^2 \right) \phi^2, \quad (3.14)$$

$$\mathcal{L}_{A\phi} = -2g (\partial_{\perp} h(x_{\perp})) A_{\perp} \phi, \quad (3.15)$$

$$\mathcal{L}_{\text{gh}} = \bar{\eta} \left( -\partial^2 + \xi g^2 h(x_{\perp})^2 \right) \eta. \quad (3.16)$$

It is particularly convenient to choose  $\xi = 1$ , because with this gauge choice the terms containing  $(\nabla A)^2$  disappear. Notice that only the  $A_{\perp}$  component of the gauge field is coupled to the  $\phi$  field, thanks to the gauge choice (3.10).

Performing the functional integral gives a product of functional determinants, as always. One can verify that with the gauge choice that we have made, the contribution of the ghosts completely cancels the contribution of the  $A_{\parallel}$  components of the gauge field.

The expression for  $\Gamma_{\text{qu}}$  is thus composed out of contributions from the Higgs boson  $\chi$  and the coupled fields  $A_{\perp}$  and  $\phi$ :

$$\Gamma_{\text{qu}}[h] = \Gamma_{\chi}[h] + \Gamma_{A\phi}[h], \quad (3.17)$$

$$\Gamma_{\chi}[h] = \ln \det \left( \frac{\Delta_{\chi}}{\Delta_{\chi}^{(0)}} \right), \quad (3.18)$$

$$\Gamma_{A\phi}[h] = \ln \det \left( \frac{\Delta_{A\phi}}{\Delta_{A\phi}^{(0)}} \right), \quad (3.19)$$

where the fluctuation operators can be read off directly from (3.12)–(3.16). Before we write them explicitly, let us introduce notation that will simplify the subsequent formulae, by defining

$$M_H(h)^2 \equiv m^2 + \frac{\lambda}{2}h^2, \quad (3.20)$$

$$M_W(h)^2 \equiv g^2h^2, \quad (3.21)$$

$$M_{A\phi}(h)^2 \equiv m^2 + \left(\frac{\lambda}{6} + g^2\right)h^2. \quad (3.22)$$

With this notation, the operator that determines quadratic fluctuation of  $\chi$  is

$$\Delta_{\chi} = -\nabla^2 + M_H(h)^2, \quad (3.23)$$

where according to (3.20)  $M_H(h)^2 = m^2 + \frac{\lambda}{2}h(x_{\perp})^2$ . The fluctuation operator that couples  $A_{\perp}$  and  $\phi$  is denoted by  $\Delta_{A\phi}$ . It is a linear operator on  $\mathcal{H}_L \oplus \mathcal{H}_L$ , and can be represented by the  $2 \times 2$  matrix:

$$\Delta_{A\phi} = \begin{pmatrix} -\nabla^2 + M_W(h)^2 & -2g(\partial_{\perp}h(x_{\perp})) \\ -2g(\partial_{\perp}h(x_{\perp})) & -\nabla^2 + M_{A\phi}(h)^2 \end{pmatrix}. \quad (3.24)$$

In (3.19),  $\Delta_{A\phi}^{(0)}$  denotes the corresponding operator in the symmetric vacuum sector.<sup>4</sup>

We can rewrite the expressions (3.18) and (3.19) using the fact that the dependence on worldvolume coordinates of these operators is trivial, and thus they can be partially diagonalized by a Fourier transform.<sup>5</sup> That is,

$$\Gamma_{\chi}[h] = \frac{V_{\parallel}}{2} \int \frac{dk_{\parallel}^{n-1}}{(2\pi)^{n-1}} \ln \frac{\det_{\perp} \tilde{\Delta}_{\chi}}{\det_{\perp} \tilde{\Delta}_{\chi}^{(0)}}, \quad (3.25)$$

$$\Gamma_{A\phi}[h] = \frac{V_{\parallel}}{2} \int \frac{dk_{\parallel}^{n-1}}{(2\pi)^{n-1}} \ln \frac{\det_{\perp} \tilde{\Delta}_{A\phi}}{\det_{\perp} \tilde{\Delta}_{A\phi}^{(0)}}, \quad (3.26)$$

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<sup>4</sup>We could equally well have chosen  $\Delta_{A\phi}^{(0)}$  to be the fluctuation operator in the Higgs phase, since the two ground states are degenerate

<sup>5</sup>This is what we also did in section 2, by rewriting  $\sigma_{\text{qu}}$  as an integral over the worldvolume momentum.

where

$$\tilde{\Delta}_\chi = k_\parallel^2 - \partial_\perp^2 + M_H(h)^2, \quad (3.27)$$

and

$$\tilde{\Delta}_{A\phi} = \begin{pmatrix} -\partial_\perp^2 + k_\parallel^2 + M_W(h)^2 & -2g(\partial_\perp h(x_\perp)) \\ -2g(\partial_\perp h(x_\perp)) & -\partial_\perp^2 + k_\parallel^2 + M_{A\phi}(h)^2 \end{pmatrix}. \quad (3.28)$$

As we saw in the previous section, the divergent integrals over the worldvolume momentum in (3.25) and (3.26) can be regularized using any convenient regulator. We just need to subtract the divergent pieces and add them back in a regularized form. We find it convenient to use dimensional regularization.

One can compute the effective potential by taking the field configuration to be coordinate-independent and going to the limit  $L, V_\parallel \rightarrow \infty$ . Then the operators  $\tilde{\Delta}_\chi$  and  $\tilde{\Delta}_{A\phi}$  can be further diagonalized, giving the following expression for the effective potential:

$$V_{\text{eff}}(\Phi) = \frac{m^2}{2}\Phi^2 + \frac{\lambda}{4!}\Phi^4 + \frac{1}{2} \int \frac{d^n k}{(2\pi)^2} \left[ \ln \left( \frac{k^2 + M_H(\Phi)^2}{k^2 + m^2} \right) + \ln \left( \frac{k^2 + M_W(\Phi)^2}{k^2} \right) + \ln \left( \frac{k^2 + M_{A\phi}(\Phi)^2}{k^2 + m^2} \right) \right]. \quad (3.29)$$

The integrals can be easily evaluated using dimensional regularization. The resulting effective potential (which has also been computed in many references, see *e.g.* Refs. [7,10]) reads

$$V_{\text{eff}}(\Phi) = \frac{m^2}{2}\Phi^2 + \frac{\lambda}{4!}\Phi^4 - \frac{1}{12\pi} \left( M_H(\Phi)^3 + M_W(\Phi)^3 + M_{A\phi}(\Phi)^3 - 2m^3 \right). \quad (3.30)$$

As we asserted earlier, a cubic  $g^3\Phi^3 = M_W(\Phi)^3$  term has appeared, leading to the relations (3.4) for the parameters of the theory.

## B. Computation of the surface tension

The energy density of a domain wall-type configuration is a sum of the classical and quantum parts, as defined in (3.8). Both classical and quantum terms immediately follow from (3.9) and (3.17), by dividing the corresponding expressions by the transverse area  $V_\parallel$ . Note, that  $\sigma_{\text{cl}}$  is now linearly divergent with increasing  $L$ , as there is no ground state degeneracy in the tree-level effective potential. This means that the quantum part of  $\sigma$  must diverge with increasing  $L$  as well, so that the total  $\sigma$  is finite as  $L \rightarrow \infty$ . Hence, we must take the  $L \rightarrow \infty$  limit at the end of the calculation, after all terms are collected together.

The quantum part,  $\sigma_{\text{qu}}$ , can be further decomposed analogously to (3.17) with corresponding terms called  $\sigma_\chi$  and  $\sigma_{A\phi}$ , respectively.

Let us concentrate on the calculation of  $\sigma_\chi$  first. We would like to apply the same technique for computing the ratio of determinants as used in the previous section. However the situation is now different, since  $v = \lim_{x_\perp \rightarrow \infty} h(x_\perp)$  is no longer the minimum of the tree-level effective potential, and thus the ratio of determinants diverges with  $L$  instead of being

asymptotically constant. In fact, it must diverge exponentially, so that after taking the logarithm, the leading divergence is linear in  $L$ .

Naturally, we would like to extract this asymptotic behavior, leaving to the computer only a calculation of a term that is finite in the limit  $L \rightarrow \infty$ . For that purpose, we impose a restriction on our field configuration. We require it to be constant outside the region  $-a \leq x_\perp \leq a$ ; so that  $h(x_\perp) = 0$  for  $x_\perp < -a$  and  $h(x_\perp) = v$  for  $x_\perp > a$ . In other words, we require the domain wall field profile to have a “thickness” no larger than  $2a$ . Eventually,  $a$  will be taken sufficiently large so that this condition has negligible effect.

Thus there now are two length parameters that we have introduced:  $a$ , being the half-length of the window where the background field is allowed to vary, and  $L$ , the half-length of the box in the direction perpendicular to the worldvolume. The quantum part of the surface tension, as well as  $\sigma_\chi$  and  $\sigma_{A\phi}$  individually, are going to be asymptotically constant in  $a$ , but linearly divergent in  $L$  as we send  $L$  to infinity.

The ratio of determinants in  $\sigma_\chi$  can be written in terms of the solution of associated ordinary differential equations, just as in Eq. (2.32). This leads to the following expression for  $\sigma_\chi$ :

$$\sigma_\chi = \frac{1}{2} \int \frac{d^{n-1}k_\parallel}{(2\pi)^{n-1}} \ln \left( \frac{\psi(L)}{\psi^{(0)}(L)} \right), \quad (3.31)$$

where  $\psi(x_\perp)$  is the solution of the differential equation

$$\left[ -\partial_\perp^2 + k_\parallel^2 + M_H(h)^2 \right] \psi(x_\perp) = 0, \quad (3.32)$$

with the boundary conditions

$$\psi(-L) = 0; \quad \psi(-L)' = 1. \quad (3.33)$$

The corresponding solution in the vacuum sector,  $\psi^{(0)}(x_\perp)$ , can be taken from (2.38):

$$\psi^{(0)}(L) = \frac{e^{(k_\parallel^2 + m^2)^{1/2} 2L}}{2(k_\parallel^2 + m^2)^{1/2}}. \quad (3.34)$$

Note that Eq. (3.31) must be rendered UV-finite by the separation and subsequent dimensional regularization of the UV-divergent piece. At this point we only note that the regularization proceeds according to the procedure established in section 2. We will return to the actual treatment of ultraviolet divergences in  $\sigma_\chi$  later.

To solve the differential equation (3.32) we divide the whole interval into three parts. Namely, region I:  $x_\perp < -a$ , region II:  $-a \leq x_\perp \leq a$ , and region III:  $x_\perp \geq a$ . We can immediately write the general solution of (3.32) in regions I and III, since  $h(x_\perp)$  restricted to these intervals is constant. Thus, we only need to integrate Eq. (3.32) in region II, and then match the solution at the intersections  $x_\perp = -a$  and  $x_\perp = a$ .

Omitting terms that fall off exponentially with increasing  $L$ , the solution in region I is given by

$$\psi(x_\perp) = \frac{1}{2(k_\parallel^2 + m^2)^{1/2}} e^{(k_\parallel^2 + m^2)^{1/2}(L+x_\perp)}, \quad x_\perp \in [-L, a] \quad (3.35)$$



The solution in region II can be written in the following form:

$$\psi(x_\perp) = \frac{f^{(II)}(x_\perp)}{2(k_\parallel^2 + m^2)^{1/2}} e^{(k_\parallel^2 + m^2)^{1/2}(L+x_\perp)}, \quad x_\perp \in [-a, a] \quad (3.36)$$

where  $f^{(II)}(x_\perp)$  satisfies the equation

$$\left[ -\partial_\perp^2 - 2(k_\parallel^2 + m^2)^{1/2}\partial_\perp + \frac{\lambda}{2}h(x_\perp)^2 \right] f^{(II)}(x_\perp) = 0, \quad x_\perp \in [-a, a] \quad (3.37)$$

with the boundary conditions

$$f^{(II)}(-a) = 1; \quad f^{(II)}(-a)' = 0. \quad (3.38)$$

The general solution in region III is

$$\psi(x_\perp) = f^{(III)} e^{(k_\parallel^2 + m^2)^{1/2}L + (k_\parallel^2 + M_H(v)^2)^{1/2}x_\perp} + g^{(III)} e^{(k_\parallel^2 + m^2)^{1/2}L - (k_\parallel^2 + M_H(v)^2)^{1/2}x_\perp}, \quad x \in [a, L] \quad (3.39)$$

Matching this solution with (3.36) by requiring the continuity of  $\psi(x_\perp)$  and its first derivative, we obtain the following expression for  $f^{(III)}$

$$f^{(III)} = \frac{e^{a[(k_\parallel^2 + m^2)^{1/2} - (k_\parallel^2 + M_H(v)^2)^{1/2}]}}{2(k_\parallel^2 + m^2)^{1/2}} \left( \frac{f^{(II)}(a)'}{(k_\parallel^2 + M_H(v)^2)^{1/2}} + f^{(II)}(a) \left[ 1 + \frac{(k_\parallel^2 + m^2)^{1/2}}{(k_\parallel^2 + M_H(v)^2)^{1/2}} \right] \right). \quad (3.40)$$

It is  $f^{(III)}$  that determines the value of  $\sigma_\chi$ , as the second term in (3.39) is exponentially suppressed in  $L$ . One can therefore take the first term in (3.39) with  $f^{(III)}$  given by (3.40) and substitute this, together with  $\psi^{(0)}(L)$  from (3.34) into the expression (3.31) for  $\sigma_\chi$ . Before we do that, however, note that for the constant background field  $h(x_\perp) \equiv v$ ,  $\psi(L)$  can be easily found analytically. This allows one to compute the part of the effective action  $\Gamma_\chi[h \equiv v]$  at finite  $L$ :

$$\frac{\Gamma_\chi[v]}{V_\parallel} = \frac{1}{2} \int \frac{d^{n-1}k_\parallel}{(2\pi)^{n-1}} \left[ 2L \left[ (k_\parallel^2 + M_H(v)^2)^{1/2} - (k_\parallel^2 + m^2)^{1/2} \right] + \frac{1}{2} \ln \left( \frac{k_\parallel^2 + M_H(v)^2}{k_\parallel^2 + m^2} \right) \right]. \quad (3.41)$$

The term that multiplies  $L$  in this expression comes from the exponent in (3.34) and the corresponding exponent in  $\psi(L)$ , while the ratio inside the log in the last term involves the coefficients that multiply the respective exponents. One can obtain part of the effective potential [the  $M_W(\Phi)^3$  term in (3.30)] by dividing the expression above by the volume of the spacetime box  $2V_\parallel L$  and taking the limit  $L \rightarrow \infty$ . The last term in (3.41) is a boundary effect reflecting our choice of Dirichlet boundary conditions. It makes no contribution to the effective potential in the  $L \rightarrow \infty$  limit.

We may require the parameters of the theory formulated in a box with finite length  $L$  to be adjusted so that the effective action evaluated at finite  $L$  vanishes at  $v$ . Of course, as

we send  $L$  to infinity, these parameters will flow to the corresponding values for the theory in infinite spacetime. This condition will prove useful momentarily. For one can notice that the part of  $\sigma_\chi V_\parallel$  that is linear in  $L$  is simply half of the first term in (3.41). Thus, one can separate  $\sigma_\chi$  as

$$\sigma_\chi = \tilde{\sigma}_\chi + \frac{\Gamma_\chi[v]}{2V_\parallel}, \quad (3.42)$$

where

$$\tilde{\sigma}_\chi = \frac{1}{2} \int \frac{d^{n-1}k_\parallel}{(2\pi)^{n-1}} \left[ \ln \left( \frac{2f^{(III)}}{(k_\parallel^2 + m^2)^{1/2}} \right) - \frac{1}{4} \ln \left( \frac{k_\parallel^2 + M_H(v)^2}{k_\parallel^2 + m^2} \right) \right] \quad (3.43)$$

is finite as  $L \rightarrow \infty$  and  $\Gamma_\chi[v]$  is given by Eq. (3.41).

Recall that  $\Gamma_\chi[v]$  is a part of the effective action evaluated at the constant field configuration  $h(x_\perp) = v$ . One can anticipate that the same story will repeat for the calculation of  $\sigma_{A\phi}$ , and  $\Gamma_{A\phi}[v]$  will be produced in a similar fashion. Thus, after we collect all parts of  $\sigma$  together, the coefficient in front of  $L$  in the resulting expression will be a multiple of  $\Gamma[v]$ , and therefore the divergence in  $L$  will vanish, leaving an expression for  $\sigma$  that is finite in the limit  $L \rightarrow \infty$ .

Having anticipated the form of the outcome, let us now compute  $\sigma_{A\phi}$ . The relevant differential operators in (3.19) are now given by  $2 \times 2$  matrices, so we must find a way to generalize the correspondence between determinants and solutions of associated differential equations. This is done in the appendix, where we show that

$$\frac{\det \tilde{\Delta}_{A\phi}}{\det \tilde{\Delta}_{A\phi}^{(0)}} = \frac{\psi_1^{(1)}(L)\psi_2^{(2)}(L) - \psi_2^{(1)}(L)\psi_1^{(2)}(L)}{\varphi_1^{(1)}(L)\varphi_2^{(2)}(L) - \varphi_2^{(1)}(L)\varphi_1^{(2)}(L)}, \quad (3.44)$$

where both  $\psi^{(1)}(x_\perp)$  and  $\psi^{(2)}(x_\perp)$  are the solutions of the same system of differential equations,

$$\tilde{\Delta}_{A\phi} \begin{pmatrix} \psi_1^{(i)}(x_\perp) \\ \psi_2^{(i)}(x_\perp) \end{pmatrix} = 0, \quad i = 1, 2, \quad (3.45)$$

with vanishing initial conditions,

$$\begin{pmatrix} \psi_1^{(i)}(-L) \\ \psi_2^{(i)}(-L) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad i = 1, 2, \quad (3.46)$$

but with differing initial derivatives,

$$\begin{pmatrix} \psi_1^{(1)'}(-L) \\ \psi_2^{(1)'}(-L) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} \psi_1^{(2)'}(-L) \\ \psi_2^{(2)'}(-L) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \quad (3.47)$$

In other words,  $\psi^{(1)}$  and  $\psi^{(2)}$  are the two linearly independent solutions of  $\tilde{\Delta}_{A\phi}\psi^{(i)} = 0$  which vanish at  $x_\perp = -L$  but which have orthogonal initial “velocities”.

The corresponding vacuum sector solutions are denoted by  $\varphi^{(i)}$ .

Hence,  $\sigma_{A\phi}$  can be written analogously to (3.31) as

$$\sigma_{A\phi} = \frac{1}{2} \int \frac{dk_{\parallel}^{n-1}}{(2\pi)^{n-1}} \ln \left( \frac{\psi_1^{(1)}(L)\psi_2^{(2)}(L) - \psi_2^{(1)}(L)\psi_1^{(2)}(L)}{\varphi_1^{(1)}(L)\varphi_2^{(2)}(L) - \varphi_2^{(1)}(L)\varphi_1^{(2)}(L)} \right). \quad (3.48)$$

The further computation of  $\sigma_{A\phi}$  proceeds similarly to that of  $\sigma_{\chi}$  in almost all respects. We again divide the interval  $[-L, L]$  into three regions, and use appropriate changes of variables to extract the asymptotic behavior of the solutions. The main results are presented below.

As in the case of  $\sigma_{\chi}$ ,  $\sigma_{A\phi}$  can be decomposed analogously to (3.42):

$$\sigma_{A\phi} = \tilde{\sigma}_{A\phi} + \frac{\Gamma_{A\phi}[v]}{2V_{\parallel}}, \quad (3.49)$$

where

$$\begin{aligned} \sigma_{A\phi} = \frac{1}{2} \int \frac{d^{n-1}k_{\parallel}}{(2\pi)^{n-1}} & \left[ \ln \left( 4 \frac{f_1^{(1,III)} f_2^{(2,III)} - f_1^{(2,III)} f_2^{(1,III)}}{(k_{\parallel}^2 + m^2)^{1/2} k_{\parallel}} \right) \right. \\ & \left. - \frac{1}{4} \ln \left( \frac{(k_{\parallel}^2 + M_W(v)^2)(k_{\parallel}^2 + M_{A\phi}(v)^2)}{k_{\parallel}^2(k_{\parallel}^2 + m^2)} \right) \right] \end{aligned} \quad (3.50)$$

and

$$\begin{aligned} \frac{\Gamma_{A\phi}[v]}{V_{\parallel}} = \frac{1}{2} \int \frac{d^{n-1}k_{\parallel}}{(2\pi)^{n-1}} & \left[ 2L \left\{ (k_{\parallel}^2 + M_W(v)^2)^{1/2} - k_{\parallel} + (k_{\parallel}^2 + M_{A\phi}(v)^2)^{1/2} - (k_{\parallel}^2 + m^2)^{1/2} \right\} \right. \\ & \left. + \frac{1}{2} \ln \left( \frac{(k_{\parallel}^2 + M_W(v)^2)(k_{\parallel}^2 + M_{A\phi}(v)^2)}{k_{\parallel}^2(k_{\parallel}^2 + m^2)} \right) \right], \end{aligned} \quad (3.51)$$

where  $f_i^{(j,III)}$  is determined analogously to  $f^{(III)}$  in (3.40). The subscript  $i$  in the former labels the component of a vector, while the superscript  $j$  distinguishes the two sets of boundary conditions. In other words,  $f_i^{(j,III)}$  comes from the  $\psi_i^{(j)}(x_{\perp})$ .

Explicit expressions for  $f_i^{(1,III)}$  are:

$$f_1^{(1,III)} = \frac{e^{a[k_{\parallel} - (k_{\parallel}^2 + M_W(v)^2)^{1/2}]}}{2k_{\parallel}} \left[ \frac{f_1^{(1,II)'}(a)}{(k_{\parallel}^2 + M_W(v)^2)^{1/2}} + f_1^{(1,II)}(a) \left( 1 - \frac{k_{\parallel}}{(k_{\parallel}^2 + M_W(v)^2)^{1/2}} \right) \right], \quad (3.52)$$

$$f_2^{(1,III)} = \frac{e^{a[k_{\parallel} - (k_{\parallel}^2 + M_{A\phi}(v)^2)^{1/2}]}}{2k_{\parallel}} \left[ \frac{f_2^{(1,II)'}(a)}{(k_{\parallel}^2 + M_{A\phi}(v)^2)^{1/2}} + f_2^{(1,II)}(a) \left( 1 - \frac{k_{\parallel}}{(k_{\parallel}^2 + M_{A\phi}(v)^2)^{1/2}} \right) \right], \quad (3.53)$$

where  $f_1^{(1,II)}(x_{\perp})$  and  $f_2^{(1,II)}(x_{\perp})$  are the solutions of the following system of ODE's,

$$\begin{pmatrix} -\partial_{\perp}^2 - 2k_{\parallel}\partial_{\perp} + M_W(h)^2 & -2g(\partial_{\perp}h) \\ -2g(\partial_{\perp}h) & -\partial_{\perp}^2 - 2k_{\parallel}\partial_{\perp} + M_{A\phi}(h)^2 \end{pmatrix} \begin{pmatrix} f_1^{(1,II)} \\ f_2^{(1,II)} \end{pmatrix} = 0, \quad (3.54)$$

subject to the boundary conditions

$$\begin{pmatrix} f_1^{(1,II)}(-a) \\ f_2^{(1,II)}(-a) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} f_1^{(1,II)'}(-a) \\ f_2^{(1,II)'}(-a) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \quad (3.55)$$

Corresponding expressions for  $f_i^{(2,III)}$  are very similar, as one might expect. The only difference is caused by the fact that  $\psi^{(1)}(x_\perp)$  and  $\psi^{(2)}(x_\perp)$  have different exponents in the region I. One finds,

$$f_1^{(2,III)} = \frac{e^{a[(k_\parallel^2+m^2)^{1/2}-(k_\parallel^2+M_W(v)^2)^{1/2}]} }{2(k_\parallel^2+m^2)^{1/2}} \left[ \frac{f_1^{(2,II)'}(a)}{(k_\parallel^2+M_W(v)^2)^{1/2}} + f_1^{(2,II)}(a) \left( 1 - \frac{(k_\parallel^2+m^2)^{1/2}}{(k_\parallel^2+M_W(v)^2)^{1/2}} \right) \right], \quad (3.56)$$

$$f_2^{(2,III)} = \frac{e^{a[(k_\parallel^2+m^2)^{1/2}-(k_\parallel^2+M_{A\phi}(v)^2)^{1/2}]} }{2k_\parallel} \left[ \frac{f_2^{(2,II)'}(a)}{(k_\parallel^2+M_{A\phi}(v)^2)^{1/2}} + f_2^{(2,II)}(a) \left( 1 - \frac{(k_\parallel^2+m^2)^{1/2}}{(k_\parallel^2+M_{A\phi}(v)^2)^{1/2}} \right) \right], \quad (3.57)$$

where  $f_1^{(2,II)}(x_\perp)$  and  $f_2^{(2,II)}(x_\perp)$  are the solutions of the following system of ODE's,

$$\begin{pmatrix} -\partial_\perp^2 - 2(k_\parallel^2+m^2)^{1/2}\partial_\perp - m^2 + M_H(h)^2 & -2g(\partial_\perp h) \\ -2g(\partial_\perp h) & -\partial_\perp^2 - 2(k_\parallel^2+m^2)^{1/2}\partial_\perp - m^2 + M_{A\phi}(h)^2 \end{pmatrix} \begin{pmatrix} f_1^{(2,II)} \\ f_2^{(2,II)} \end{pmatrix} = 0, \quad (3.58)$$

subject to the boundary conditions

$$\begin{pmatrix} f_1^{(2,II)}(-a) \\ f_2^{(2,II)}(-a) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} f_1^{(2,II)'}(-a) \\ f_2^{(2,II)'}(-a) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \quad (3.59)$$

Now one can collect together  $\sigma_{\text{cl}}$ ,  $\sigma_\chi$  and  $\sigma_{A\phi}$ , and in particular combine the terms which add up to make  $\Gamma[v]/2V_\parallel$ , and set it to zero. At this point we need to recall our prescription for regularization, which was introduced in section 2. Namely, we need to subtract the divergent pieces from  $\sigma_\chi$  and  $\sigma_{A\phi}$ , and then add them back, with the divergent loop diagrams replaced by their regularized values.

Naively, the divergent term in  $\sigma_\chi$ , for example, is given by

$$\sigma_\chi^{(div)} = \frac{\lambda}{4} G \text{tr} [(\nabla^2 + m^2)^{-1}], \quad (3.60)$$

where

$$G = \int_{-L}^L dx_\perp h^2(x_\perp) \quad (3.61)$$

is defined analogously to  $F$  in Eq. (2.29), and the trace involves integration over  $k_\parallel$  and summing over discrete  $x_\perp$  modes that vanish at  $\pm L$ . However, when extracting  $\Gamma[v]/2V_\parallel$ , we should also add and subtract the corresponding UV-divergent term which is just

$$\frac{\lambda}{4} L v^2 \operatorname{tr} [(\nabla^2 + m^2)^{-1}] = \frac{\lambda}{4} \left( \int_0^L dx_\perp v^2 \right) \operatorname{tr} [(\nabla^2 + m^2)^{-1}]. \quad (3.62)$$

Therefore, the UV-subtraction in  $\tilde{\sigma}_\chi$  should actually involve (3.60) with  $G$  replaced by

$$\tilde{G} = \int_{-L}^L dx_\perp [h^2(x_\perp) - \theta(x_\perp) v^2]. \quad (3.63)$$

Here  $\theta(x_\perp)$  is the step function which is zero for  $x_\perp < 0$  and one for  $x_\perp > 0$ . Note that  $\tilde{G}$  is finite in the limit  $L \rightarrow \infty$ .

The treatment of ultraviolet divergences in  $\tilde{\sigma}_{A\phi}$  proceeds in a similar manner.

Taking the  $L \rightarrow \infty$  limit we arrive at an expression for  $\sigma$  that only involves finite integrals that can be computed numerically:

$$\sigma = \lim_{L \rightarrow \infty} [\tilde{\sigma}_{\text{cl}} + \tilde{\sigma}_\chi + \tilde{\sigma}_{A\phi}], \quad (3.64)$$

where

$$\tilde{\sigma}_{\text{cl}} = \int dx_\perp \left\{ \frac{1}{2} (\partial_\perp h(x_\perp))^2 + \frac{m^2}{2} [h(x_\perp)^2 - \theta(x_\perp) v^2] + \frac{\lambda}{4!} [h(x_\perp)^4 - \theta(x_\perp) v^4] \right\}, \quad (3.65)$$

$$\begin{aligned} \tilde{\sigma}_\chi &= \frac{1}{2} \int \frac{d^2 k_\parallel}{(2\pi)^2} \left[ \ln \left( \frac{2f^{(III)}}{(k_\parallel^2 + m^2)^{1/2}} \right) - \frac{1}{4} \ln \left( \frac{k_\parallel^2 + M_H(v)^2}{k_\parallel^2 + m^2} \right) - \frac{\lambda}{4} \frac{\tilde{G}}{(k_\parallel^2 + m^2)^{1/2}} \right] \\ &\quad + \frac{\lambda}{4} \frac{\tilde{G}}{4\pi} m, \end{aligned} \quad (3.66)$$

$$\begin{aligned} \tilde{\sigma}_{A\chi} &= \frac{1}{2} \int \frac{d^2 k_\parallel}{(2\pi)^2} \left[ \ln \left( 4 \frac{f_1^{(1,III)} f_2^{(2,III)} - f_1^{(2,III)} f_2^{(1,III)}}{(k_\parallel^2 + m^2)^{1/2} k_\parallel} \right) \right. \\ &\quad \left. - \frac{1}{4} \ln \left( \frac{(k_\parallel^2 + M_W(v)^2)(k_\parallel^2 + M_W(v^2))}{k_\parallel^2 (k_\parallel^2 + m^2)} \right) - \frac{g^2}{2} \frac{\tilde{G}}{k_\parallel} - \left( \frac{\lambda}{12} + \frac{g^2}{2} \right) \frac{\tilde{G}}{(k_\parallel^2 + m^2)^{1/2}} \right] \\ &\quad + \left( \frac{\lambda}{12} + \frac{g^2}{2} \right) \frac{\tilde{G}}{4\pi} m. \end{aligned} \quad (3.67)$$

Note that the “classical” term (3.65) is also finite as  $L \rightarrow \infty$ , as it is defined by the same subtraction that was needed to isolate  $\Gamma_{\text{cl}}[v]$ ,

$$\tilde{\sigma}_{\text{cl}} = \sigma_{\text{cl}} - \frac{\Gamma_{\text{cl}}[v]}{2V_\parallel}. \quad (3.68)$$

The terms outside the integrals in  $\tilde{\sigma}_\chi$  and  $\tilde{\sigma}_{A\phi}$  are the values of UV-divergent terms of the type (3.60), computed in the limit of infinite  $L$  by dimensional continuation.

### C. Numerical evaluation

It is now possible, by using Eq's. (3.64)–(3.67), to compute the full one-loop surface tension for an arbitrary domain wall-type field configuration. One just needs to solve the

relevant differential equations (3.37), (3.54), and (3.58), and then perform a final integration over the worldvolume momentum  $k_{\parallel}$ . The latter step merely involves a one-dimensional integration over the radial component of  $k_{\parallel}$ , as the integrands in  $\sigma_{\chi}$  and  $\sigma_{A\phi}$  depend only on  $k_{\parallel}^2$ .

We have written a simple *Mathematica* program to implement these steps. It allowed computation of  $\sigma$  with a relative accuracy of 0.2% or better.<sup>6</sup> As described below, we tested the numerical results against perturbative calculations when  $\lambda/g^2 \ll 1$ , and obtained good agreement.

In principle, one may calculate the one-loop energy of the true domain wall configuration by minimizing (3.64) with respect to the domain wall profile. This we have not bothered to do.

#### D. Comparison with perturbative $\lambda/g^2$ expansion

As noted earlier, for our specific example of the Abelian Higgs model, the ratio of scalar to gauge couplings,  $\lambda/g^2$ , functions as the loop expansion parameter controlling the reliability of perturbation theory at the phase transition. For asymptotically small values of  $\lambda/g^2$  one may analytically compute the leading and first subleading correction to the surface tension  $\sigma$ . One way to do this is to start with the derivative expansion of the effective action:

$$\Gamma = \int d^3x \left[ V_{\text{eff}}(h) + \frac{1}{2} Z(h) (\partial_{\mu} h)^2 + \dots \right], \quad (3.69)$$

where  $\dots$  stands for the terms with four or more derivatives. We denote the first two terms in this expansion by  $\Gamma^{(2)}$ .

To leading order in  $\eta \equiv \lambda/g^2$  one may take the wavefunction renormalization factor  $Z(h)$  in the effective action (3.69) to equal one. The reason for this is that the one-loop corrections to the classical kinetic term in (3.69) behave like  $g^4 h^2 M_W(h)^{-3} (\partial h)^2$  and  $\lambda^2 h^2 M_H(h)^{-3} (\partial h)^2$ , and hence their contribution to  $\sigma$  is suppressed in comparison with the leading term by a factor of  $g^2/M_W(v) \sim \eta$  and  $\lambda/M_H(v) \sim \eta^{3/2}$ , respectively. (Recall, from (3.4), that  $M_W(v) = gv \sim g^4/\lambda$  while  $M_H(v) \sim m \sim g^3/\sqrt{\lambda}$ .) Hence, the leading term in the expansion of  $\sigma$  is completely determined by the leading term in the expansion of the one-loop effective potential in powers of  $\eta$ . Recall that for a scalar theory with a canonical kinetic term and any effective potential with two degenerate minima at  $\Phi = 0$  and  $\Phi = v$ , normalized so that  $V_{\text{eff}}(0) = V_{\text{eff}}(v) = 0$ , the domain wall field configuration is a solution of the classical equations of motion whose conserved “energy”  $E \equiv \frac{1}{2} (\partial_{\perp} h)^2 - V_{\text{eff}}(h)$  vanishes. Consequently, the surface tension, neglecting the wavefunction renormalization and all higher derivative terms, is given by

$$\sigma = \int_{-\infty}^{\infty} dx_{\perp} (\partial_{\perp} h)^2 = \int_0^v dh \sqrt{2V_{\text{eff}}(h)}. \quad (3.70)$$

---

<sup>6</sup>Higher accuracy is undoubtedly feasible given more care than we used in the numerical implementation. This is not completely trivial however, as Eq’s. (3.54) and (3.58) can be quite stiff due to the presence of two different length scales set by the mass parameters  $M_W(h)$  and  $M_{A\phi}(h)$ .

To obtain the leading behavior of the integral (3.70), one may expand the one-loop potential (3.30) for small  $\eta$ , obtaining just the classic double well form (3.3). Inserting this into Eq. (3.70) gives

$$\sigma = \sigma^{(0)} (1 + \mathcal{O}(\eta)), \quad (3.71)$$

with

$$\sigma^{(0)} = \frac{m}{v} \int_0^v dh h(v-h) = \frac{1}{6} m v^2 = \frac{2}{\sqrt{3}\pi} m^2 \eta^{-3/2}. \quad (3.72)$$

[The final form uses the leading behavior of  $v^2 = \frac{4\sqrt{3}}{\pi} \eta^{-3/2} m (1 + \mathcal{O}(\eta))$ .] To get the complete next-to-leading term, it is necessary to calculate both the two-loop effective potential and the one-loop correction to the wavefunction renormalization, *i.e.*,  $Z(h) - 1$ . One can not go beyond the next-to-leading order term using the derivative expansion, since the Higgs field contributions to the higher order terms in the expansion (3.69) give corrections to the surface tension that all scale like  $\sigma^{(0)} \eta^{3/2}$ . To see this, one may look, for example, at the contributions to the two- and four-derivative terms that behave like  $\lambda^2 h^2 M_H(h)^{-3} (\partial h)^2$  and  $\lambda^2 M_H(h)^{-5} (\partial h)^4$ , respectively. The first term represents a  $Z$ -factor correction and scales like  $\sigma^{(0)} \eta^{3/2}$ . The second term is further suppressed by a factor of  $[l M_H(v)]^{-2}$  where  $l$  is the characteristic thickness of domain wall. But this is  $\mathcal{O}(M_H(v)^{-1})$  and so this four derivative contribution also scales like  $\sigma^{(0)} \eta^{3/2}$ . It is easy to see that further higher order derivative terms also scale like  $\sigma^{(0)} \eta^{3/2}$ .

More generally, the expansion of  $\sigma$  in powers of  $\eta$  breaks down because of infrared problems [9]. This can be seen by looking at the contribution of the gauge field to the four derivative term computed to one-loop order. This term behaves like  $g^4 M_W(h)^{-5} (\partial h)^4$  and by naive power counting gives a contribution to  $\sigma$  that scales as  $\sigma^{(0)} \eta^2$ . However, since  $h(x_\perp)$  vanishes exponentially in the symmetric phase (*i.e.*, as  $x_\perp \rightarrow -\infty$ ), and  $M_W(h) = gh$ , the integral over  $x_\perp$  in the effective action diverges.

In fact, there is another, more significant source of infrared divergences. One can see this by expanding the effective potential to higher orders in  $\eta$ . One finds that the integral which enters the  $\mathcal{O}(\eta^{3/2})$  term is logarithmically divergent in the infrared. In particular, this implies that the next-to-next-to-leading order correction to  $\sigma$  is  $\mathcal{O}(\eta^{3/2} \log \eta)$ .

Nevertheless for small  $\lambda/g^2$ , one might hope that the one-loop value of  $\sigma$  could be well approximated by the contributions from the effective potential plus the first wavefunction renormalization correction, while neglecting all higher order terms in the derivative expansion. It was argued in Ref. [9] that this is indeed the case in electroweak theory, at least for intermediate values of the Higgs masses. Unfortunately the infrared divergences of the individual terms in the various expansions used in [9] complicate the accurate computation of  $\sigma$ . In contrast, the technique that we use in this paper does not have infrared problems, as it effectively resums all one-loop corrections to the surface tension, allowing its accurate computation for any value of  $\lambda/g^2$ .

In what follows we show that the inclusion of the wavefunction renormalization gives results that are very close to the full one-loop value of  $\sigma$  given by (3.64), as long as  $\lambda/g^2$  is less than 1. The comparison with analytic results also provides a nice consistency check of Eq. (3.64).

To make this comparison, it is necessary to calculate the wavefunction renormalization in the particular gauge (3.10). We use the convenient method of Ref. [14] to determine the first two terms in the derivative expansion of the effective action (3.69), which we denote by  $\Gamma^{(2)}$ . According to Ref. [14], the one-loop contribution to  $\Gamma^{(2)}$  can be written as

$$\delta\Gamma_{\text{one-loop}}^{(2)} = \delta\Gamma_{\chi}^{(2)} + \delta\Gamma_{A\phi}^{(2)}, \quad (3.73)$$

where

$$\delta\Gamma_{\chi}^{(2)} = \frac{1}{2} \int d^3x \int \frac{d^3k}{(2\pi)^3} \left[ \ln \frac{\hat{\Delta}_{\chi}}{\hat{\Delta}_{\chi}^{(0)}} + \frac{k^2}{3} \left( \partial_{\mu} \hat{\Delta}_{\chi}^{-1} \right)^2 \right], \quad (3.74)$$

and

$$\delta\Gamma_{A\phi}^{(2)} = \frac{1}{2} \int d^3x \int \frac{d^3k}{(2\pi)^3} \left[ \text{tr} \ln \frac{\hat{\Delta}_{A\phi}}{\hat{\Delta}_{A\phi}^{(0)}} + \frac{k^2}{3} \text{tr} \left( \partial_{\mu} \hat{\Delta}_{A\phi}^{-1} \right)^2 \right]. \quad (3.75)$$

In the equations above, the trace runs over matrix indices and the hat on top of the operators  $\Delta_{\chi}$  and  $\Delta_{A\phi}$  [defined in Eq's. (3.23) and (3.24)] denotes Fourier transform with respect to  $k$ . For example,

$$\hat{\Delta}_{\chi} = k^2 + M_H(h)^2. \quad (3.76)$$

The wavefunction renormalization

$$Z(h) = 1 + \delta Z_{\chi} + \delta Z_{A\phi} \quad (3.77)$$

receives contributions both from  $\delta\Gamma_{\chi}^{(2)}$  and  $\delta\Gamma_{A\phi}^{(2)}$ , denoted by  $\delta Z_{\chi}$  and  $\delta Z_{A\phi}$ , respectively. The scalar contribution can be easily evaluated:

$$\delta Z_{\chi} = \frac{\lambda^2 h^2}{3} \int \frac{d^3k}{(2\pi)^3} \frac{k^2}{(k^2 + M_H(h)^2)^4} = \frac{\lambda^2 h^2}{M_H(h)^3} \frac{1}{192\pi}. \quad (3.78)$$

To calculate  $\delta\Gamma_{A\phi}^{(2)}$  one should notice that contributions with two derivatives come from both terms in Eq. (3.75), as the fluctuation operator (3.24) already contains the derivative term  $g(\partial h)$ . A computation which is analogous to the one done in [9] leads to the result,

$$\delta Z_{A\phi} = -\frac{g^2}{M_W(h) + M_{A\phi}(h)} \frac{1}{\pi} + \left( \frac{g^4 h^2}{M_W(h)^3} + \frac{(\frac{\lambda}{6} + g^2)^2 h^2}{M_{A\phi}(h)^3} \right) \frac{1}{48\pi}. \quad (3.79)$$

One can easily check that  $\delta Z_{A\phi} \sim \eta$  and  $\delta Z_{\chi} \sim \eta^{3/2}$ , in accord with our earlier discussion.

The expression for the surface tension which takes into account the  $Z$ -factor is a simple generalization of (3.70). The vanishing “energy” now implies that  $\frac{1}{2}Z(h)(\partial_{\perp} h)^2 = V_{\text{eff}}(h)$ , and expression (3.70) becomes

$$\sigma = \int_{-\infty}^{\infty} dx_{\perp} Z(h)(\partial_{\perp} h)^2 = \int_0^v dh \sqrt{2Z(h)V_{\text{eff}}(h)}. \quad (3.80)$$

We can compute the first two terms in the expansion of this  $Z$ -factor corrected  $\sigma$  by plugging in expressions (3.77)–(3.79) for  $Z(h)$ , and (3.30) for  $V_{\text{eff}}(h)$ , rescaling the field as  $h = v\tilde{h}$  to



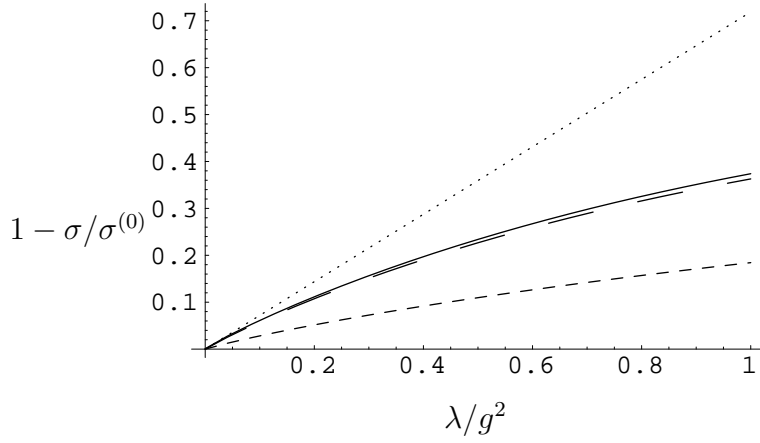


FIG. 1. Comparison of the full one-loop surface tension (3.64) [solid line], the second order derivative expansion result (3.80) computed with the full one-loop effective potential and wavefunction renormalization [long dashes], the “classical” result (3.70) computed with the full one-loop effective potential [short dashes], and the first two terms of the small  $\eta$  asymptotic expansion (3.84) [dotted line].

extract a factor of  $v$ , and expanding the integrand in powers of  $\eta$ . The leading term in the expansion is just  $\sigma^{(0)}$ . To get the subleading term it is necessary to use values of  $g$  and  $v$  accurate to next-to-leading order:

$$g = 3^{1/4} \sqrt{\pi} \eta^{1/4} m^{1/2} \left( 1 - \frac{3}{32} \eta + \mathcal{O}(\eta^{3/2}) \right), \quad (3.81)$$

$$v = \left( \frac{48}{\pi^2} \right)^{1/4} \eta^{-3/4} m^{1/2} \left( 1 - \frac{1}{32} \eta + \mathcal{O}(\eta^{3/2}) \right). \quad (3.82)$$

One finds that the first subleading term is given by

$$\sigma^{(1)} = \frac{m^2}{8\sqrt{3}\pi} \eta^{-1/2} \int_0^1 d\tilde{h} (\tilde{h} - 1)(17 + 18\tilde{h}) = \frac{23}{16\sqrt{3}\pi} m^2 \eta^{-1/2}. \quad (3.83)$$

Hence, the  $Z$ -factor corrected surface tension (3.80) has the small  $\eta$  expansion

$$\sigma = \sigma^{(0)} \left[ 1 + \frac{23}{32} \eta + \mathcal{O}(\eta^{3/2} \log \eta) \right], \quad (3.84)$$

with  $\sigma^{(0)}$  given in Eq. (3.72).

The result (3.84) is not the complete next-to-leading order surface tension, as two-loop corrections to the effective potential also contribute to the subleading  $\mathcal{O}(\eta)$  term. Including these two-loop contributions is straightforward, but will not be examined here.

It is interesting to compare the different approximations to the surface tension given by (i) the full one-loop result (3.64), (ii) the expression (3.80) computed with the complete one-loop effective potential (3.30) and wavefunction renormalization factor (3.77), (iii) the expression (3.70) computed with the complete one-loop effective potential (3.30), and (iv) the first two terms (3.84) of the small  $\eta$  asymptotic expansion. All these expressions omit two (and higher) loop contributions to the effective potential, while (ii) also omits four

derivative (and higher) contributions to the one-loop effective action, and (iii) omits all one-loop derivative corrections to the effective action. This comparison is shown in Fig. 1. The full one-loop result (3.64) was evaluated using the lowest order profile for the domain wall,

$$h(x_{\perp}) = \frac{v}{2} [\tanh(mx_{\perp}/2) + 1], \quad (3.85)$$

with  $v$  being the true (not just leading order) value of the scalar VEV in the Higgs phase. Corrections to the profile of the domain wall only affect  $\sigma$  at next-to-next to leading order in the small  $\eta$  expansion (3.84). (This is tantamount to using the unperturbed wavefunction to compute the first order correction to the energy of a state in quantum mechanics.)

From Fig. 1 it is clear that difference between the second order derivative expansion result (3.80) and the full one-loop result is quite small, at least for  $\lambda/g^2 \leq 1$ . Numerically, this difference appears to scale as  $\eta^{3/2}$ . In contrast, the “classical” result (3.70) which omits all one-loop derivative corrections, and the leading two terms of the asymptotic expansion (3.84), both depart significantly from the full one-loop result for quite small values of  $\eta$ .

#### IV. CONCLUSION

The method presented in this paper provides a straightforward technique for evaluating the one-loop energy density of any configuration which depends non-trivially on only a single coordinate. It allows ultraviolet divergent contributions to be clearly isolated and extracted using any convenient regulator (which does not constrain momenta in the one non-trivial direction). As shown by our example of scalar QED, the method remains applicable in gauge theories, where multiple fluctuating fields are unavoidably coupled. Generalizing the treatment to, for example, full electroweak theory is straightforward.

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#### APPENDIX A: FUNCTIONAL DETERMINANTS FROM THE SOLUTIONS OF ASSOCIATED DIFFERENTIAL EQUATIONS

The ratio of one-dimensional determinants (2.32) follows from the result that, for any bounded functions  $U^{(0)}(x_{\perp})$  and  $U^{(1)}(x_{\perp})$ ,

$$\frac{\det_{\perp} (-\partial_{\perp}^2 + U^{(0)}(x_{\perp}) - \lambda)}{\det_{\perp} (-\partial_{\perp}^2 + U^{(1)}(x_{\perp}) - \lambda)} = \frac{\psi(\lambda; L)}{\varphi(\lambda; L)}, \quad (A1)$$

where  $\psi$  and  $\varphi$  are the solutions to the associated ordinary differential equations,

$$[-\partial_{\perp}^2 + U^{(0)}(x_{\perp}) - \lambda] \psi(\lambda; x_{\perp}) = 0, \quad (A2)$$

$$[-\partial_{\perp}^2 + U^{(1)}(x_{\perp}) - \lambda] \varphi(\lambda; x_{\perp}) = 0, \quad (A3)$$

with boundary conditions specifying vanishing initial values

$$\psi(\lambda; -L) = 0; \quad \varphi(\lambda; -L) = 0, \quad (\text{A4})$$

and unit initial slopes,

$$\psi(\lambda; -L)' = 1; \quad \varphi(\lambda; -L)' = 1. \quad (\text{A5})$$

Here primes denote  $\partial/\partial x_\perp$ , and the functional determinants in (A1) are defined on the space  $\mathcal{H}_L$  of functions vanishing at  $x_\perp = \pm L$ .

To prove the result (A1), one should need only notice that both LHS and RHS of this equation are meromorphic functions of  $\lambda$  with the same poles and zeroes, and both sides approach one as  $|\lambda| \rightarrow \infty$  in any direction away from the positive real axis. Consequently they must be identical. This proof is due to Coleman (see Ref. [12]). It may be easily generalized to the case of higher-dimensional operators. Let  $W^{(a)} = \|W_{ij}^{(a)}\|$ ,  $a = 0, 1$ , be two self-adjoint second order differential operators acting on  $\bigoplus_{i=1..n} \mathcal{H}_L$  of the form  $W_{ij}^{(a)} = -\partial_\perp^2 + U_{ij}^{(a)}(x_\perp)$ . Then by the same argument sketched above,

$$\frac{\det_\perp(W^{(1)} - \lambda I)}{\det_\perp(W^{(0)} - \lambda I)} = \sum_{i_1 \dots i_n} \frac{\epsilon_{i_1 \dots i_n} \psi_{i_1}^{(1)}(\lambda; L) \dots \psi_{i_n}^{(n)}(\lambda; L)}{\epsilon_{i_1 \dots i_n} \varphi_{i_1}^{(1)}(\lambda; L) \dots \varphi_{i_n}^{(n)}(\lambda; L)}, \quad (\text{A6})$$

where  $I$  is the unit matrix, and  $\psi$ 's and  $\varphi$ 's are the solutions of the associated system of ODEs,

$$\sum_j (W_{ij}^{(1)} - \lambda \delta_{ij}) \psi_j^{(k)}(\lambda; x_\perp) = 0, \quad (\text{A7})$$

$$\sum_j (W_{ij}^{(0)} - \lambda \delta_{ij}) \varphi_j^{(k)}(\lambda; x_\perp) = 0, \quad (\text{A8})$$

with the initial conditions

$$\psi_i^{(k)}(\lambda; -L) = 0; \quad \psi_i^{(k)}(\lambda; -L)' = \delta_{ik}, \quad (\text{A9})$$

$$\varphi_i^{(k)}(\lambda; -L) = 0; \quad \varphi_i^{(k)}(\lambda; -L)' = \delta_{ik}. \quad (\text{A10})$$

The solutions  $\{\psi^{(1)}(x_\perp), \dots, \psi^{(n)}(x_\perp)\}$  are linearly independent, and therefore the numerator in the RHS of (A6) vanishes if and only if there is a non-trivial solution of  $W^{(1)}\psi(\lambda; x_\perp) = \lambda\psi(\lambda; x_\perp)$  which vanishes at  $x_\perp = \pm L$ . In other words, the numerators of both the left and right hand sides of Eq. (A6) vanish whenever  $\lambda$  is an eigenvalue of the linear operator  $W^{(1)}$  (defined on the interval  $[-L, L]$  with Dirichlet boundary conditions). The same argument applies to the denominators on either side. One may also show that both sides approach unity as  $|\lambda| \rightarrow \infty$  in any direction away from the positive real axis, and therefore conclude that both sides must be equal.

The specific relation (3.44) immediately follows as the special case of  $n = 2$ ,  $W^{(1)} = \tilde{\Delta}_{A\phi}$ ,  $W^{(0)} = \tilde{\Delta}_{A\phi}^{(0)}$ , and  $\lambda = 0$ .

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